Search for integrable 4-D accelerator mappings

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Abstract
We model an accelerator lattice by a mapping consisting of a 4x4 symplectic matrix followed by a thin non-linear lens of special properties. Our goal is to find such non-linear lenses that can eliminate chaotic trajectories in a large volume of the beam phase-space while also having large betatron tune spreads. In this report we present several attempts (so far of limited success) to find such lattices.

1. Introduction

By the term "4-d integrable mapping" we imply a mapping which possesses 2 independent integrals of motion in involution[2], such that these integrals of motion are analytic functions of phase-space variables, x, p_x, y, p_y. In principle, a 4-D mapping possessing a single integral of motion is also of interest for various reasons. Recently, one of the authors (V.D.) has proposed [1] several practical solutions for employing 2-D integrable nonlinear lattices in accelerators. His approach relies on using the so-called 2-D McMillan mapping [2], which is known to be integrable (non-chaotic). In this paper we report on our search for an integrable 4-D mapping, which has the following two steps. First, a linear transformation,

$$\begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{y} \\ \tilde{p}_y \end{pmatrix} = M \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}, \tag{1}$$

where M is a symplectic 4x4 matrix, which is followed by a thin lens,

$$\tilde{p}_x = \tilde{p}_x + f(\tilde{x}, \tilde{y})$$
$$\tilde{p}_y = \tilde{p}_y + g(\tilde{x}, \tilde{y}) \tag{2}$$

For this 4-D mapping to be symplectic, the lens functions must possess the following property:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \tag{3}$$

One can see that this property is attained if

$$f = \frac{\partial \varphi}{\partial x} \text{ and } g = \frac{\partial \varphi}{\partial y}. \tag{4}$$

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[2] By "involution" we mean that such integrals have zero Poisson bracket.

Since we are interested in practical accelerator mappings, the potential function, φ, should also satisfy the Laplace equation, $\Delta\varphi = 0$, which implies that

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 . \tag{5}$$

Equations (3) and (5) define special properties of the thin lens in Eq. (2).

In this report we are limiting our search for 4-D integrable mappings to the following type of the linear matrix M:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\frac{1}{\beta} & 0 \end{pmatrix}, \tag{6}$$

where β is a non-zero coefficient.

2. Example of integrable 4-D accelerator map

Consider the following 2-D magneto-static potential $\varphi(x,y)$:

$$\varphi(x,y) = \mathrm{Re}\left[\frac{d}{2a}\ln\left(az^2+1\right)\right], \tag{7}$$

where $z = x+iy$, and d and a are scaling coefficients.

One can check by inspection that this potential satisfies the Laplace equation: $\Delta\varphi(x,y) = 0$, and thus can be created by appropriately placed currents. For example, in a vacuum, such a potential can be created by two parallel linear currents, located at $y = \pm(a)^{-1/2}$.

Recalling that the magnetic field $\vec{H} = curl(\vec{A})$, where (in this case) $\vec{A} = (0,0,\varphi)$, we have

$$H_x = \frac{\partial\varphi}{\partial y} \text{ and } H_y = -\frac{\partial\varphi}{\partial x}. \tag{8}$$

Consider a charged particle traveling in vacuum and having the following Cartesian momentum components:

$$\vec{p} = \begin{pmatrix} p_x & p_y & p_s \end{pmatrix} \text{ with } p_s >> p_x, p_y .$$

Suppose now that this particle enters a region with the magnetic field components as in Eq. (8). If this region is short enough, the particle will experience a delta-function like kick, δp, determined by the equation of motion:

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\left[\vec{v}\times\vec{H}\right]. \tag{9}$$

The linearized unitless kicks (neglecting end effects) are:

$$\delta p_x = H_y = -\frac{\partial\varphi}{\partial x} = -\mathrm{Re}\left(\frac{dz}{az^2+1}\right), \tag{10}$$

$$\delta p_y = -H_x = -\frac{\partial\varphi}{\partial y} = \mathrm{Im}\left(\frac{dz}{az^2+1}\right). \tag{11}$$

Let us introduce a complex momentum variable, $p = p_x + ip_y$. Then, the momentum kicks in Eqs. (10) and (11) can be presented as:

$$\delta p = \delta p_x + i\,\delta p_y = -\frac{d\bar{z}}{a\bar{z}^2 + 1}, \tag{12}$$

where $\bar{\ }$ represents a complex conjugate. Now consider the following mapping:

$$z_{n+1} = \bar{p}_n,$$
$$p_{n+1} = -\bar{z}_n - \frac{d\bar{z}_{n+1}}{a\bar{z}_{n+1}^2 + 1}. \tag{13}$$

This mapping is equivalent to a mapping defined by Eqs. (1) and (2) with matrix M being

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{14}$$

and the non-linear lens functions f and g being

$$f = -\frac{d\left(ax^3 + ay^2x + x\right)}{a^2x^4 + 2a^2x^2y^2 + a^2y^4 + 2ax^2 - 2ay^2 + 1}, \tag{15}$$

$$g = -\frac{d\left(ay^3 + ax^2y - y\right)}{a^2x^4 + 2a^2x^2y^2 + a^2y^4 + 2ax^2 - 2ay^2 + 1}. \tag{16}$$

It has been demonstrated before by one of the authors (V.D.) that the mapping (13) is integrable. It has the following complex integral of motion:

$$I = az^2\bar{p}^2 + z^2 + \bar{p}^2 + dz\bar{p}. \tag{17}$$

One can show that the Re and Im parts of this integral are independent analytic functions in involution:

$$I_1 = \mathrm{Re}(I) = a\left(x^2 - y^2\right)\left(p_x^2 - p_y^2\right) + 4axyp_xp_y + x^2 - y^2 + p_x^2 - p_y^2 + dxp_x + dyp_y \tag{18}$$

$$I_2 = \mathrm{Im}(I) = 2axy\left(p_x^2 - p_y^2\right) - 2a\left(x^2 - y^2\right)p_xp_y + 2xy - 2p_xp_y - dxp_y + dyp_x \tag{19}$$

If the motion is one-dimensional with either $y = p_y = 0$ or $x = p_x = 0$, the second integral, I_2, becomes zero while the first integral becomes a familiar McMillan-type integral:

$$I_x = ax^2p_x^2 + x^2 + p_x^2 + dxp_x \tag{20}$$

or

$$I_y = ay^2p_y^2 - y^2 - p_y^2 + dyp_y. \tag{21}$$

The mapping (14, 15, 16) (while being quite interesting from dynamics point of view) has one essential fault from the accelerator point of view. To form the fields described by (15) and (16) one will have to use linear currents or magnetic poles with spacing of the order of $\pm(a)^{-1/2}$. These current conductors or pole-pieces will create a natural physical boundary for outer-most particle trajectories. However, the nature of integrals (18) and (19) is such that particle amplitudes for all particles around origin grow, far exceeding the boundary set by $\pm(a)^{-1/2}$. In essence, the small-amplitude motion is unstable while it remains finite and integrable. Figure 1 shows an example of a particle trajectory for $a = 1$ and $d = 1$.

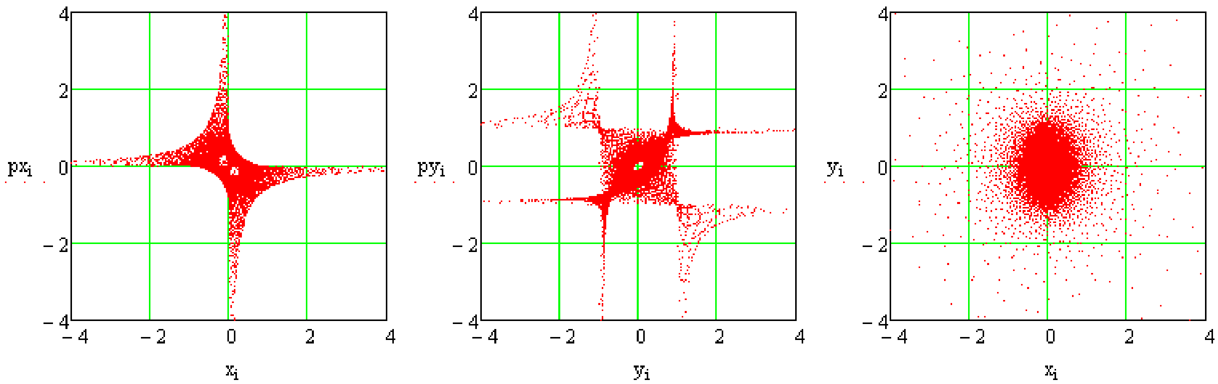

Figure 1: The mapping (14, 15, 16) for 10000 iterations with a = 1 and d = 1, and the initial conditions $x = 0.2$, $p_x = 0.1$, $y = 0.2$, and $p_y = 0.2$.

Despite the problem with this mapping, the fact that it is integrable encouraged us to look for other mappings acceptable from accelerator point of view. In what follows we concentrate on our attempts to find integrable mappings for $\beta \neq \pm 1$ (see Eq. 6).

3. Modified 4-D accelerator mapping.

It appears that the problems of the mapping (14, 15, 16) can be circumvented by changing the beta-function (coefficient β) in one of the planes. Clearly, if $\beta \to 0$, the non-linear lens kick becomes essentially one-dimensional and the motion nearly integrable in 4-D with one of the integrals given by Eq. (20) and another corresponding to the Courant-Snyder invariant for the linear portion of the mapping. A variation of this method was proposed in Ref. [1]. On the other hand, we have observed that even modest deviations of β from ± 1 lead to the appearance of regular bound trajectories. Figure 2 shows an example of mapping with $\beta = 0.8$ but otherwise identical to that in Fig. 1.

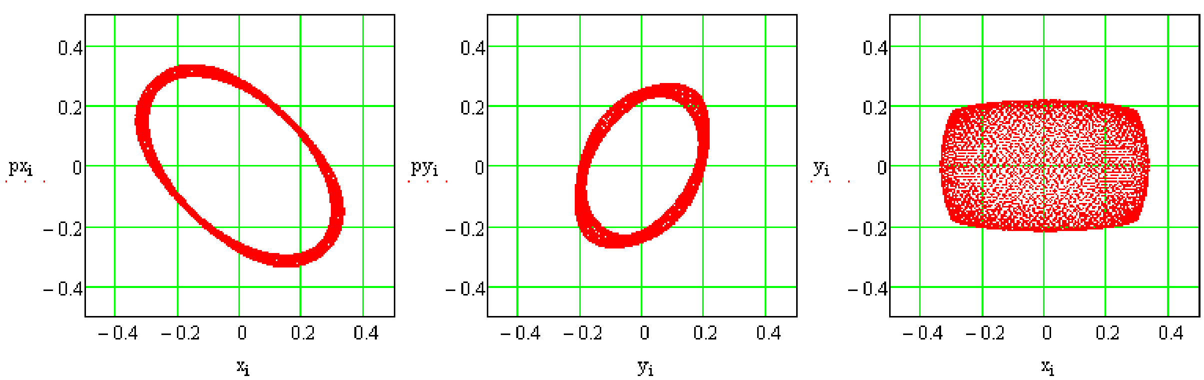

Figure 2: The mapping from Fig. 1 for 10000 iterations with $\beta = 0.8$, a = 1 and d = 1, and the initial conditions $x = 0.2$, $p_x = 0.1$, $y = 0.2$, and $p_y = 0.2$.

One can also observe that the one-dimensional motion (y=p_y=0 and x=p_x=0) also possesses an integral of motion given by:

$$I_x = ax^2 p_x^2 + x^2 + p_x^2 + dxp_x \tag{22}$$

and

$$I_y = ay^2 p_y^2 - \frac{y^2}{\beta^2} - p_y^2 + dyp_y \,. \tag{23}$$

Thus, it is reasonable to assume that the mapping with an arbitrary value of β might have at least one integral of motion. In the next section we report on our direct search of this integral using a polynomial substitution. In section 5 we examine the trajectories numerically in search of chaotic ones.

4. Direct search of polynomial integrals

Our first hypothesis was that integrals, if they exist, could be represented in a polynomial form, like Eq. (18) and (19). Thus, an attempt was made to find polynomial integrals by direct testing. In this case one should compare the polynomial substitution on two successive steps, such that if it is an integral of motion, it should remain constant for arbitrary initial conditions.

We have considered the following polynomial of the *n*-th order:

$$P_n(V) = \sum_{\substack{i,j,k,l \geq 0 \\ i+j+k+l \leq n \\ i+j+k+l > 0}} C_{i,j,k,l} x^i p_x^j y^k p_y^l \,, \tag{24}$$

where $V = (x, p_x, y, p_y)$. If there is such a combination of $C_{i,j,k,l}$ that $P_n(V) = P_n(\widetilde{V})$ for any V, than one can obtain at least one integral of motion by equating coefficients C for equal powers of coordinates and momenta.

To obtain such $C_{i,j,k,l}$, one can demand that all coefficients of the polynomial:

$$(P_n(V) - P_n(\widetilde{V}))(a^2x^4 + 2a^2x^2y^2 + a^2y^4 + 2ax^2 - 2ay^2 + 1)^n \,, \tag{25}$$

vanish. This leads to a system of linear equations, the solution of which could provide us with integral(s) of motion.

A code was written in Mathematica to perform described steps. No solutions with polynomials of the 6-th order were found in the case of $\beta \neq 1$ in (6). In the case of $\beta = 1$ integrals (18) and (19) were obtained to verify the search algorithm.

5. Application of SALI to detect chaotic trajectories

The Smaller Alignment Index (SALI) method was proposed in Ref. [3] to distinguish rapidly and reliably between ordered and chaotic motion in 4-D symplectic mappings. For each initial condition, the method is to compute the SALI for each iteration. For large number of iterations (10^4 – 10^5), two different scenarios for SALI behavior are possible: (1) it remains non-zero for ordered orbits or (2) it decays exponentially to zero for chaotic orbits. For our first example, Figure 3 shows SALI behavior for mapping in Fig. 1. Since the mapping is integrable, there are no chaotic orbits in the entire 4-D phase space.

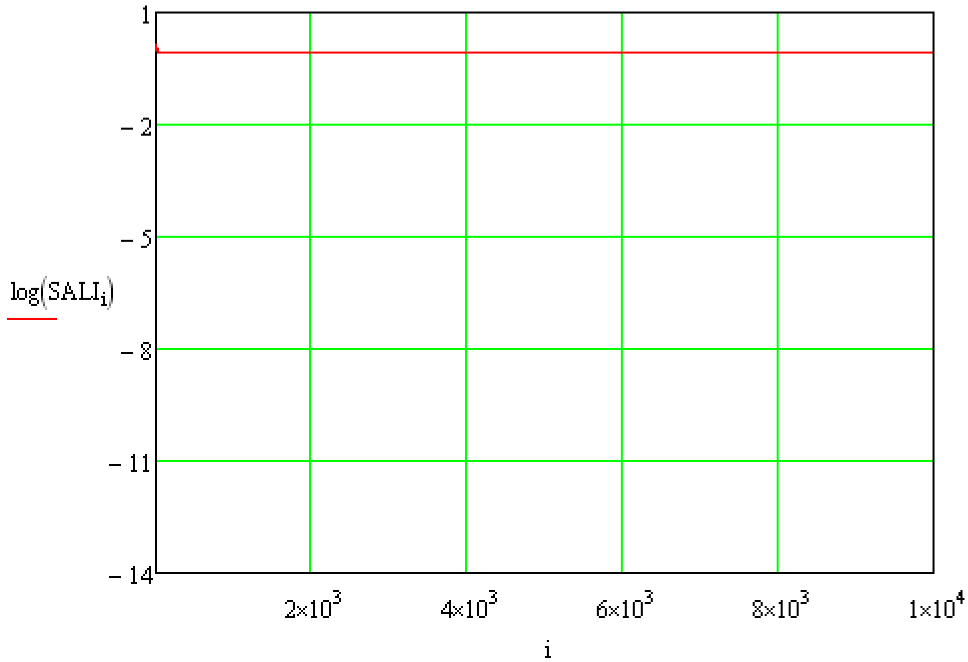

Figure 3: Evolution of SALI for the trajectory presented in Fig. 1.

While examining SALI behavior for various initial conditions corresponding to the mapping in Fig.2, we have observed that both chaotic and ordered orbits exist for such a mapping. Figure 4 presents the evolution of SALI for two different initial conditions, corresponding to chaotic and ordered orbits.

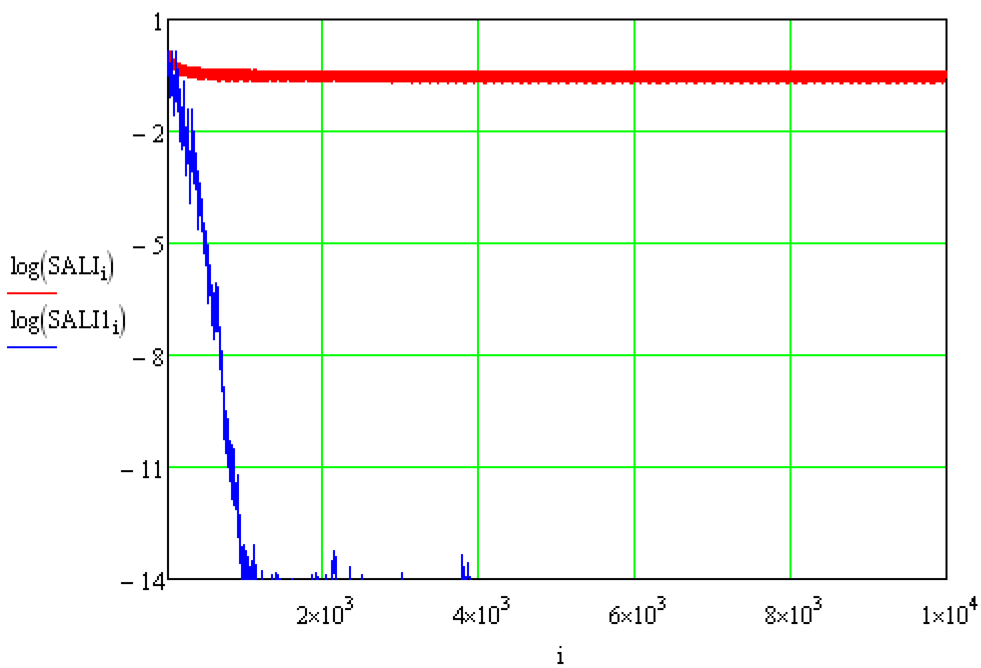

Figure 4: Evolution of SALI for the ordered trajectory presented in Fig. 2 (top, red) and for a chaotic trajectory (bottom, blue) with the same initial conditions except $y = 0.6$.

The presence of chaotic trajectories with $\beta \neq \pm 1$ is an indication that the mapping is not integrable. However, it may still have a single integral of motion.

6. Conclusions

It is clear to us that integrable 4-D accelerator-type mappings exist, as evidenced by our example in Sec. 2. All known to us mappings of that type have a fault which we were unable to overcome yet, while retaining the mapping integrability. The mapping in section 3 may be useful if one can prove that it has a single integral of motion, which would guarantee the absence of Arnold diffusion.

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